JOURNAL OF ALGEBRA 5, 106-109 (1967)

On Groups without Abelian Composition Factors

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Communicated by Marshall Hall, Jr.

Received January 14, 1966

INTRODUCTION

It is well known (see, e.g., [*1*], p. 131) that a minimal normal subgroup N of a finite group G is the direct product of conjugates of a simple group H. Denoting the automorphism group of H by $A(H)$ and the inner automorphism group by $I(H)$ we assume the following:

(*) *H is non-Abelian and $I(H)$ has a complement in $A(H)$.*

We show that in this case N has a complement C in G, and that G is what Neumann [*2*] has called a twisted wreath product of H by C. We use this result to characterize those finite groups, all of whose composition factors H satisfy (*), as iterated twisted wreath products of simple groups satisfying (*).

1. CONSTRUCTION OF C

Let $C(H)$ be a complement for $I(H)$ in $A(H)$. For $x \in G$, $S \subseteq G$, we denote $x^{-1}Sx$ by S^x. By a well known result which occurs, for example, as a problem in Reference 1, p. 135, the automorphisms of N permute the direct factors of N. Thus, for any $x \in G$, H^x occurs as a direct factor of N, and N is the direct product of the groups H^{t_i}, where the t_i are coset representatives of the normalizer of H in G. For convenience we take $t_1 = 1$. We denote by $\bar{x}$ the permutation of the direct factors of N induced by conjugation by x. Thus

$$H^{t_i x} = H^{t_{\bar{x}(i)}}$$

Let $\alpha_i(x)$ be the automorphism of H induced by conjugation by $t_i x t_{\bar{x}(i)}^{-1}$. Let $C = \{x \in G \mid \alpha_i(x) \in C(H), \text{all } i\}$.

THEOREM 1. *C is a complement for N in G.*

Proof. Clearly $1 \in C$ holds since $\alpha_i(1)$ is the identity automorphism for each i.

Since $\overline{xy}$ (i) $= \bar{y}(\bar{x}(i))$, we have for $x, y \in C$ that $\alpha_i(xy)$ is induced by

$$t_i(xy)t_{\overline{xy}(i)}^{-1} = (t_i x t_{\bar{x}(i)}^{-1})(t_{\bar{x}(i)} y t_{\bar{y}(\bar{x}(i))}^{-1}).$$

Thus $\alpha_i(xy) = \alpha_i(x)\alpha_{\bar{x}(i)}(y) \in C(H)$. C is therefore a subgroup of G.

Let $x \in N \cap C$. Since N normalizes the H^{t_i}, $\bar{x}$ is the identity permutation. $\alpha_i(x)$ is therefore induced by $t_i x t_i^{-1}$ which belongs to N.

Let $t_i x t_i^{-1} = \prod h_j^{t_j}$. Since $h_j^{t_j}$ centralizes H for $j \neq 1$, $\alpha_i(x)$ is the inner automorphism of H induced by h_1. Thus from $x \in C$ we conclude that $\alpha_i(x) \in I(H) \cap C(H) = 1$ for each i. x thus centralizes all direct factors of N. Since N is easily seen to have trivial center, it follows that $x = 1$. We have shown that N and C have trivial intersection and it remains only to show that their product is the whole group G.

Since H satisfies (*), for $x \in G$ we may write the automorphisms $\alpha_i(x)$ as products $\gamma_i(x)\beta_i(x)$, where $\beta_i(x) \in I(H)$ and $\gamma_i(x) \in C(H)$. Let h_i be an element of H which induces the inner automorphism $\beta_i(x)$. We put

$$z = \prod (h_j^{-1})^{t_{\bar{x}(j)}} \qquad \text{and} \qquad y = xz.$$

Clearly $\bar{y} = \bar{x}$, since N normalizes its own direct factors. Thus $\alpha_i(y)$ is induced by

$$t_i y t_{\bar{y}(i)}^{-1} = t_i(xz)t_{\bar{x}(i)}^{-1} = (t_i x t_{\bar{x}(i)}^{-1})(t_{\bar{x}(i)} z t_{\bar{x}(i)}^{-1}),$$

from which we see that

$$\alpha_i(y) = \alpha_i(x)\alpha_{\bar{x}(i)}(z).$$

But since $H^{t_{\bar{x}(j)}}$ centralizes $H^{t_{\bar{x}(i)}}$ for $i \neq j$, $\alpha_{\bar{x}(i)}(z)$ is induced by h_i^{-1}, hence equals $[\beta_i(x)]^{-1}$; from which it follows that

$$\alpha_i(y) = \gamma_i(x), \qquad \text{which is in} \quad C(H).$$

Since i was arbitrary we conclude that $y \in C$, and that $x = yz^{-1}$ belongs to CN.

2. Proof that G is a Twisted Wreath Product of H by C

We begin by recalling the definition of the twisted wreath product. Given groups A and B, a subgroup S of B with coset representatives T, and a homomorphism α from S to the automorphism group of A, we give a group structure to the set F of functions from T into A by putting $fg(t) = f(t)g(t)$ for $t \in T$. F is thus the direct product of k copies of A, where k is the index of S in B. With $b \in B$ we associate the function $\bar{b}$ from F to F defined as follows:

$$(\bar{b}(f))(t) = (\alpha(s^{-1}))(f(t')),$$

where t' and s are determined by the equation $tb^{-1} = st'$. Neumann [2] has shown that $\bar{b}$ is an automorphism of F, and the the map which associates $\bar{b}$ with b is a homomorphism from B into the automorphism group of F. The twisted wreath product of A by B is the splitting extension (semidirect product) of F by B, where the action of $b \in B$ on F is given by $\bar{b}$. Neumann has shown that this product is independent of the choice of coset representatives for S.

Since N normalizes H and C is a complement for N, we may assume without loss of generality that the coset representatives t_i for the normalizer of H in G lie in C. We take $\bar{G}$ to be the twisted wreath product of H by C with respect to the normalizer of H in C with coset representatives t_i, where for x in the normalizer of H in C, α is defined by

$$(\alpha(x))(h) = x^{-1}hx.$$

THEOREM 2. *G is isomorphic to $\bar{G}$.*

Proof. G is the semidirect product of N by C ([1], p. 89) and $\bar{G}$ is the semidirect product by C of a group naturally isomorphic to N. It is necessary therefore only to verify that the conjugation action of C corresponds under this isomorphism to the action of C specified in the wreath product definition. It suffices to show that for $c \in C$,

$$c^{-1}\left(\prod h_i^{t_i}\right)c = \prod_i h_{\bar{i}}^{s_i^{-1}t_i}$$

where $t_i c^{-1} = s_i t_{\bar{i}}$. But

$$\prod_i h_{\bar{i}}^{s_i^{-1}t_i} = \prod_i h_{\bar{i}}^{t_{\bar{i}}c} = c^{-1}\left(\prod_i h_{\bar{i}}^{t_{\bar{i}}}\right)c = c^{-1}\left(\prod h_i^{t_i}\right)c$$

since the distinct conjugates of H centralize each other.

3. GROUPS WITHOUT ABELIAN COMPOSITION FACTORS

DEFINITION. *G is an iterated twisted wreath product of $G_1, \ldots, G_n$ if there exist $\bar{G}_1, \ldots, \bar{G}_n$ such that*

(i) *G_1 is isomorphic to $\bar{G}_1$.*
(ii) *$\bar{G}_{i+1}$ is a twisted wreath product of G_{i+1} by $\bar{G}_i$ for $i = 1, \ldots, n-1$.*
(iii) *G is isomorphic to $\bar{G}_n$.*

THEOREM 3. *If every composition factor of a finite group G satisfies* (*), *then G is an iterated twisted wreath product of simple groups.*

Proof. We proceed by induction on the order of G. Let N be a minimal normal subgroup of G and H a simple direct factor of N. Since H is a composition factor of G it must satisfy (*). By theorem 1, N has a complement C in G. Since C is isomorphic to G/N, its composition factors are composition factors of G, hence satisfy (*). By induction C is an iterated twisted wreath product of simple groups. By theorem 2, G is the twisted wreath product of the simple group H by C, and the theorem is proved.

We remark that it is easily seen that an iterated twisted wreath product of finite simple groups which satisfy (*) has precisely these groups as composition factors. Thus finite groups whose composition factors satisfy (*) are characterized as iterated twisted wreath products of simple groups which satisfy (*).

COROLLARY. *If every composition factor of the finite group G is an alternating group of degree 5 or of degree at least 7, then G is an iterated twisted wreath product of alternating groups of those degrees.*

Proof. It is well-known ([3], p. 314) that the automorphism group of such an alternating group is the corresponding symmetric group. Any transposition will therefore provide a complement for the inner automorphism group, which, of course, is the alternating group itself.

We close by mentioning that G. Pandya has shown that the alternating group on six letters does not satisfy (*).

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